tions for arbitrary small strain rates is not physical, because a different model of the medium is required in this case.

Moreover, in the general case it is required to solve the appropriate boundary-value problem, in which case appreciable changes can be incurred. The conclusions obtained above can only be valid for domains in which the characteristic space scale is much greater than the space scale of the unstable perturbations.

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A TECHNIQUE FOR THE SOLUTION OF WAVE PROBLEMS FOR A NONLINEAR COMPRESSIBLE MEDIUM

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In this article we give analytical solutions of the problems of one-dimensional and two-dimensional stationary shock propagation in an ideal nonlinearly compressible medium under the action of sudden strong disturbances in the form of explosive impulses. We investigate the one-dimensional nonstationary problems of a plane and a spherical layer, and in the two-dimensional context solve the problem of the action of a moving disturbance (load) on an inelastic half-plane for the case in which the velocity D of a disturbance moving along its boundary is greater than the shock propagation velocity in the material of the half-plane. It is assumed in all the problems that the medium at the shock is subjected instantaneously to a nonlinear load and that linear irreversible loading takes place in the disturbed postshock region (Fig. 1). This statement of the problems permits them to be solved by the inverse route, i.e., by specifying a definite form (velocity) for the shock surface and determining the corresponding loading profile at the boundary of the layer or half-plane. In this case the motion of the medium in the unloading region is described by the wave equation in two variables, in application to which a Cauchy problem is formulated; it is known [1] that a solution of this problem exists and is unique. In a concrete example we examine the case in which the equation for the shock surface is given as a second-degree polynomial in ξ and we compare the results of the computations with results obtained on the basis of the method of characteristics [2], which yields satisfactory agreement of all the parameters of the medium.

The case of linear loading and unloading of the medium for the two-dimensional problem has been investigated in [3, 4]. A solution of the problem of the propagation of convergent spherical and cylindrical shocks in an ideal inelastic medium with rigid unloading is given in [5]. The investigated problems have potential practical applications in the study of strong disturbances in water-impregnated soils and in reservoirs.

§1. Propagation of One-Dimensional Plane and Spherical Shocks in a Nonlinearly Compressible Medium

Let a monotonically decreasing load $p_0(t)$ be applied to the boundary of a layer. As a result, a shock wave propagates in the medium with leading edge $r=R(t)$, behind which unloading takes place. In this case, for the disturbed region we have equations of motion, continuity, and state in the form

$$\frac{\partial u}{\partial t}+\frac{1}{\rho}\frac{\partial p}{\partial r}=0,\quad \frac{\partial \rho}{\partial t}+\rho\left(\frac{\partial u}{\partial r}+\frac{\nu u}{r}\right)=0,$$
$$p(r,t)=p^*+E(\varepsilon-\varepsilon^*), \tag{1.1}$$

where $\varepsilon=1-\rho_0/\rho$, $E=c_p^2\rho$. At the shock $r=R(t)$ we have relations of the form

$$u^*(t)=\varepsilon^*\dot{R},\ \ p^*=\rho^*\varepsilon^*\dot{R}^2,\ \ p^*(t)=\alpha_1\varepsilon^*+\alpha_2\varepsilon^{*2}\ \ (\dot{R}=dR/dt). \tag{1.2}$$

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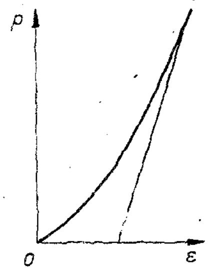

Fig. 1

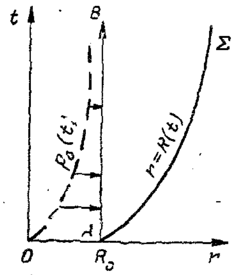

Fig. 2

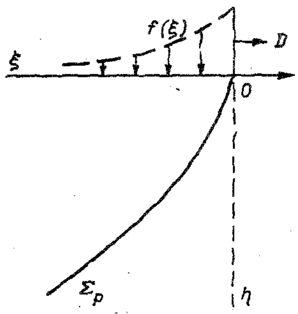

Fig. 3

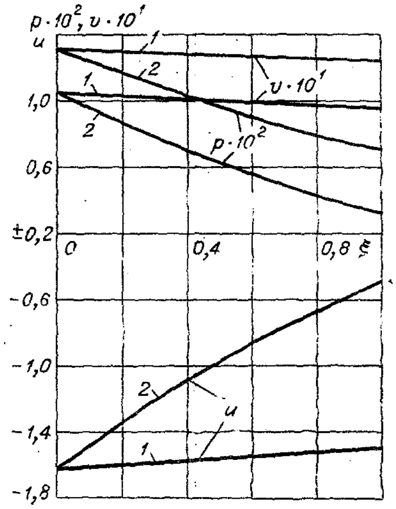

Fig. 4

We augment the system (1.1), (1.2) with the boundary condition at $r = R_0$

$$p = p_0(t). \tag{1.3}$$

Here u is the particle velocity; ρ is the density; p is the pressure; ε is the volume strain; $\nu = 0$, 2 refers to a plane and a spherical layer, respectively; and the parameters referring to the shock are denoted by an asterisk.

In the case of a plane one-dimensional wave, i.e., for $\nu = 0$, from the system (1.1) we deduce the equation

$$\frac{\partial^2 u}{\partial t^2} - c_p^2 \frac{\partial^2 u}{\partial r^2} = 0, \tag{1.4}$$

which has a solution of the form

$$u(r, t) = f_1(r - c_p t) + f_2(r + c_p t), \tag{1.5}$$

where the unknown functions $f_i(\xi)$ $(i = 1, 2)$ are determined from the boundary condition (1.3) with regard for (1.2). As mentioned, we solve this problem by an inverse method, i.e., consider the law of propagation of the shock $r = R(t)$ to be given. Then all parameters of the medium, including $u^*(t)$ and $\varepsilon^*(t)$ at the shock surface Σ, with (1.2) taken into account, are known and have the form

$$\varepsilon^*(t) = \frac{\left(1 - \frac{\alpha_1}{\alpha_2}\right)}{2} - \sqrt{\frac{\left(1 - \frac{\alpha_1}{\alpha_2}\right)^2}{4} - \frac{\rho_0 \dot{R}^2 - \alpha_1}{\alpha_2}},$$
$$u^*(t) = \dot{R}(t)\varepsilon^*(t). \tag{1.6}$$

Thus, for the wave equation (1.4), in the sector BAΣ (Fig. 2) we obtain a modified Cauchy problem with specified parameters (1.6) on the curve AΣ, in which the first equation essentially takes the place of the condition for the gradient of the velocity $u^*(t)$. Then from (1.6), taking (1.1) and (1.5) into account, we obtain the following expressions for the determination of f_1' and f_2':

$$f_1'[R(t)-c_p t]=-\frac{\ddot{R}}{2c_p}\left[\Delta_1(t)+\frac{\frac{\rho_0}{\alpha_2}c_p\dot{R}}{\Delta_2(t)}\right],$$
$$f_2'[R(t)+c_p t]=\frac{\ddot{R}}{2c_p}\left[\Delta_1(t)-\frac{\frac{\rho_0}{\alpha_2}c_p\dot{R}}{\Delta_2(t)}\right], \tag{1.7}$$

where

$$\Delta_1(t)=\frac{\left(1-\frac{\alpha_1}{\alpha_2}\right)}{2}-\Delta_2(t);$$
$$\Delta_2(t)=\sqrt{\frac{\left(1-\frac{\alpha_1}{\alpha_2}\right)^2}{4}-\frac{\rho_0\dot{R}^2-\alpha_1}{\alpha_2}}.$$

Now taking (1.7) into account, substituting (1.5) into the first equation of (1.1), and integrating with respect to r from $r=R_0$ to $r=R(t)$, for the load $p_0(t)$ we obtain

$$p_0(t)=p^*(t)+\frac{\rho_0}{2}\left\{\int_{R_0}^{R(t)}\ddot{R}[F(z_1)]\left[\Delta_1(F(z_1))+\frac{\frac{\rho_0}{\alpha_2}c_p\dot{R}(F(z_1))}{\Delta_2(F(z_1))}\right]dr+\right.$$
$$\left.+\int_{R_0}^{R(t)}\ddot{R}[F(z_2)]\left[\Delta_1(F(z_2))-\frac{\frac{\rho_0}{\alpha_2}c_p\dot{R}(F(z_2))}{\Delta_2(F(z_2))}\right]dr\right\},$$

where $z_{1,2}=r\mp c_p t$ and $F(z_{1,2})$ is the root of the equation $R(t)\mp c_p t=z_{1,2}$ with respect to the time t.

In the case of a spherical wave, i.e., for $\nu=2$, we obtain from (1.1)

$$\frac{\partial^2 u}{\partial t^2}-c_p^2\frac{\partial^2 u}{\partial r^2}-\frac{2c_p^2}{r}\left(\frac{\partial u}{\partial r}-\frac{u}{r}\right)=0,$$

which admits a solution of the form

$$u(r,t)=\frac{\psi'(r-c_p t)+\Phi'(r+c_p t)}{r}-\frac{\psi(r-c_p t)+\Phi(r+c_p t)}{r^2}, \tag{1.8}$$

where the prime signifies differentiation with respect to the argument.

After certain transformations and allowance for (1.8), from (1.6) we deduce the following expressions for the determination of $\psi''(z_1)$ and $\Phi''(z_2)$:

$$\psi''(z_1)=\psi''(R_0)+\int_{R_0}^{z_1}\Phi(\xi_1)\,d\xi_1,$$
$$\Phi''(z_2)=-\psi''[R(F(z_2))-c_pF(z_2)]+2\dot{R}(F(z_2))\Delta_1(z_2)-\frac{\rho_0}{\alpha_2}\frac{\dot{R}(F(z_2))\ddot{R}(F(z_2))}{\Delta_2(z_2)}, \tag{1.9}$$

TABLE 1

ξ	u		v		p	
	I	II	I	II	I	II
0	—1,644	—1,644	13,100	13,100	105	105
0,1	—1,628	—1,628	13,025	13,020	103,956	103,937
0,2	—1,610	—1,613	12,944	12,940	102,921	102,979
0,3	—1,597	—1,597	12,861	12,860	101,896	101,958
0,4	—1,581	—1,581	12,780	12,780	100,882	100,937
0,5	—1,565	—1,566	12,699	12,700	99,880	99,978
0,6	—1,550	—1,551	12,621	12,620	98,888	99,021
0,7	—1,535	—1,535	12,543	12,540	97,904	98,000
0,8	—1,519	—1,520	12,466	12,470	96,928	97,042
0,9	—1,505	—1,505	12,390	12;390	95,966	96,085
1,0	—1,490	—1,490	12,314	12,320	95,009	95,127

where

$$\Phi(z_1) = -\frac{\dot{R}^3(F(z_1))\,\Delta_1(F(z_1))}{2c_pR(\dot{R}-c_p)}\left[1+\frac{R\ddot{R}}{\dot{R}^2}\left(6+\frac{R\dddot{R}}{\dot{R}\,\ddot{R}}\right)\right]-$$

$$-\frac{\frac{\rho_0}{\alpha_2}\dot{R}^3\ddot{R}}{2c_p(\dot{R}-c_p)\,\Delta_2(F(z_1))}\left[2+\frac{R\ddot{R}}{\dot{R}^2}-\frac{c_p(\dot{R}+c_p)}{\dot{R}^2}\right]-$$

$$-\frac{\frac{\rho_0}{\alpha_2}R\dot{R}\,\ddot{R}^2}{2c_p(\dot{R}-c_p)\,\Delta_2^3(F(z_1))}\left\{\Delta_2^2(F(z_1))\left[2+\frac{(2\dot{R}+c_p)}{\dot{R}}\left(1+2\frac{\dot{R}^2}{R\ddot{R}}+\right.\right.\right.$$

$$\left.\left.\left.+\frac{\dot{R}\dddot{R}}{\ddot{R}^2}\right)\right]+\frac{\rho_0}{\alpha_2}\dot{R}(2\dot{R}+c_p)\right\};$$

$\psi'(R_0)$, $\psi''(R_0)$, $\Phi'(R_0)$ are determined from conditions (1.2) as $t \to 0$.

The derivation of an expression for the load $p_0(t)$ in the case of a spherical wave with allowance for (1.9) is analogous to the case of a plane wave.

§ 2. Propagation of a Two-Dimensional Wave in a Nonlinearly Compressible Medium

We investigate the planar problem of motion of a monotonically decreasing load with supersonic velocity D along the boundary of a half-plane, the material of which is modeled by an ideal medium having nonlinear and plastic properties (see Fig. 1). Then a shock with a curvilinear surface Σ_p will propagate in the half-plane, the medium at the shock is loaded by assumption, and unloading takes place behind the shock. In this case, at the surface Σ_p we obtain the following from the condition of conservation of mass and momentum:

$$\rho_0 a = \rho^*(a - v_n^*), \quad \rho_0 a v_n^* = p^*, \quad v_\tau^* = 0 \quad (a = D\sin\alpha). \tag{2.1}$$

Inasmuch as the loading profile is assumed to remain invariant in the course of shock propagation, the problem is stationary, and in the unloading zone we have the following equations in the moving coordinate system $\xi = Dt + x$, $\eta = y$:

$$D\frac{\partial u}{\partial\xi}+\frac{1}{\rho}\frac{\partial p}{\partial\xi}=0, \quad D\frac{\partial v}{\partial\xi}+\frac{1}{\rho}\frac{\partial p}{\partial\eta}=0,$$
$$D\frac{\partial\rho}{\partial\xi}+\rho\left(\frac{\partial u}{\partial\xi}+\frac{\partial v}{\partial\eta}\right)=0. \tag{2.2}$$

The boundary condition has the form

$$\text{for } \eta = 0, \text{ and } \xi \geqslant 0, p = f(\xi), \tag{2.3}$$

where $f(\xi)$ is a known monotonically decreasing function; v_τ^*, v_n^* are the tangential and normal components of the velocity of the medium toward the shock Σ_p; u, v are the projections of the velocity onto the ξ and η axes; and α is the angle of inclination of the shock surface Σ_p relative to the boundary of the half-plane.

To solve the problem we substitute the first equation of (2.2) into the third. Then for the velocity potential φ we obtain the wave equation

$$\mu^2\frac{\partial^2\varphi}{\partial\xi^2}-\frac{\partial^2\varphi}{\partial\eta^2}=0 \quad \left(\mu^2=\frac{D^2}{c_p^2}-1\right), \tag{2.4}$$

which for $D > c_p$ has a solution of the form

$$\varphi(\xi, \eta) = f_3(\xi - \mu\eta) + f_4(\xi + \mu\eta).$$

If a definite configuration is imparted to Σ_p, then the velocity components u, v of the medium are represented as follows for $\eta = \eta(\xi)$ with regard for (2.1):

$$u=\frac{\partial\varphi}{\partial\xi}=-D\sin^2\alpha(\xi)\left[\frac{\rho_0D^2}{\alpha_2}\sin^2\alpha(\xi)-\frac{\alpha_1}{\alpha_2}\right],$$
$$v=\frac{\partial\varphi}{\partial\eta}=D\sin\alpha(\xi)\cos\alpha(\xi)\left[\frac{\rho_0D^2}{\alpha_2}\sin^2\alpha(\xi)-\frac{\alpha_1}{\alpha_2}\right], \tag{2.5}$$

where $\eta(\xi)$ is the equation for the shock surface Σ_p. Hence, in the case of a two-dimensional wave, inside the curvilinear sector $\xi O \Sigma_p$ (Fig. 3) for (2.4) in conjunction with (2.5), as in Sec. 1, we obtain a Cauchy problem, and for the determination of $f_i(z_i)$ we have the expressions

$$f_i(z_i) = \mp \frac{D}{2\mu} \int_0^{z_i} \frac{\mathrm{tg}\,\alpha\,[F_i(z_i)]\{1 \pm \mu\,\mathrm{tg}\,\alpha\,[F_i(z_i)]\}\,\Phi_i(z_i)}{\{1 + \mathrm{tg}^2\,\alpha\,[F_i(z_i)]\}^2}\, dz_i, \tag{2.6}$$

in which

$$\Phi_i(z_i) = \left(\frac{\rho_0 D^2}{\alpha_2} - \frac{\alpha_1}{\alpha_2}\right)\mathrm{tg}^2\,\alpha\,[F_i(z_i)] - \frac{\alpha_1}{\alpha_2};$$

$F_i(z_i)$ $(i=3, 4)$ is the root of the equation $\xi \mp \mu\eta(\xi) = z_i$ with respect to ξ, the upper sign being taken in (2.6) in the case $i=3$. We point out that in the inverse statement of the problem, i.e., with the shock surface specified, condition (2.3) provides an expression for the determination of the loading profile $f(\xi)$.

Thus, on the basis of (2.5) and (2.6) we have obtained a solution for the problem of propagation of a two-dimensional nonlinear wave in a half-plane. If we substitute that solution into (2.3), in principle we should obtain a decaying loading profile with a sharp leading edge at the coordinate origin and unloading of the medium should take place in the disturbed zone.

An analysis of the velocity and pressure equations obtained above, together with the results of computations, shows that the unloading process can be achieved if the shock velocity decays with depth into the half-plane.

We note that a similar inverse approach has been applied to the problem of an unloading wave [6].

To illustrate the method we consider the case in which Σ_p is specified by a second-degree polynomial, i.e.,

$$\eta(\xi) = \mathrm{tg}\,\alpha_0\,\xi - \frac{b}{2}\xi^2. \tag{2.7}$$

The results of calculations by the analytical method with regard for (2.7) for $\tan\alpha_0 = 0.1255$, $b = 0.86 \cdot 10^{-3}$ and by the method of characteristics [2] are summarized in Table 1, in which I refers to the numerical method of characteristics and II to the analytical method. Figure 4 gives curves of the pressure and velocity along the shock Σ_p along the boundary of the half-plane for the case $b=0.86 \cdot 10^{-3}$ (curve 1), $0.86 \cdot 10^{-2}$ (curve 2). It is evident from Table 1 that the results obtained by both methods exhibit satisfactory agreement and the loading profile $f(\xi)$ found by the inverse method is monotonically decreasing along ξ. It is noted in Fig. 4 that the pressure p^* and the velocity components u^*, v^* along the shock Σ_p decay linearly with depth into the half-plane; in the case $b=0.86 \cdot 10^{-2}$ the decay of the indicated quantities is more rapid than for $b=0.86 \cdot 10^{-3}$. The computations show that all the parameters of the medium, including the pressure for $\eta=0$ along ξ (at the boundary of the half-plane), decay differently as a function of the values of the coefficient b. In the case $b = 0.86 \cdot 10^{-2}$ this process turns out to be more rapid and nonlinear. Hence, if the shock velocity decays comparatively rapidly with depth into the half-plane, the parameters of the medium, the pressure in particular, will also decay rapidly along the boundary of the half-plane. But the process of decay of the parameters of the medium along the boundary $\eta = 0$ is faster than at the shock.

In summary, we have presented an inverse analytical method for the solution of one- and two-dimensional stationary problems in the case of strong impulsive disturbances with regard for the nonlinear plastic strain of an ideal inelastic medium. In the case $\alpha_2 = 0$ the results of Sec. 2 agree with the results obtained by Kapustyanskii [4] on the basis of the Mellin transform.

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STRUCTURE OF ELASTIC DISCONTINUITIES ON SHOCK PROFILES IN VISCOELASTIC MEDIA

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UDC 534.222.22

Godunov and Kozin [1] have investigated the structure of shock waves in a viscoelastic medium characterized by a tangential-stress relaxation time τ and a special type of elastic energy equation for the medium. The present author [2] has formulated inequality-type constraints for the elastic energy, which are sufficient conditions for the shock-structure problem with a specified particle velocity and a structure of the form in [1] to have a unique solution. The indicated shock structure has the singular attribute that for wave velocities greater than the velocity of sound the wave profile suffers a discontinuity, which corresponds to a jump-type variation in the elastic constants of the medium. Mathematical relations at the discontinuity have been derived in [1] from heuristic considerations, and to justify the selected type of relations it is necessary to formulate a small dissipation mechanism that would produce shock "smearing" by an amount of the order of the characteristic dissipation scale. The dissipation mechanism introduced in the present study is viscous friction with a small viscosity coefficient μ; it is shown that under the constraints imposed in [2] on the equations of state positive μ corresponds to a unique continuous solution of the shock-structure problem. It is also shown that as μ tends to zero, the solution of the "shock-smearing" problem tends to the solution of the problem in [1, 2]. Inasmuch as these tendencies are nonuniform in the case of supersonic shock types, the limit solution contains a mathematical discontinuity satisfying the shock relations [1].

§1. Conditions on the Elastic Energy Equation

Let us consider a homogeneous isotropic medium with internal energy density per unit mass given by the equation

$$E = E(\alpha,\ \beta,\ \gamma,\ S), \tag{1.1}$$

in which E is a symmetric function of the parameters α, β, γ, which represent the logarithms of the relative elongations k_1, k_2, k_3 along the principal strain axes, and S is the entropy density per unit mass. Following [2], we assume that expression (1.1) satisfies the inequalities

$$T = \frac{\partial E}{\partial S} > 0, \quad r = \left(\frac{\partial E}{\partial \alpha} - \frac{\partial E}{\partial \beta}\right)\Big/(\alpha - \beta) > 0; \tag{1.2}$$

$$c^2 = \frac{\partial^2 E}{\partial \alpha^2} - \frac{\partial E}{\partial \alpha} > 0, \quad l = \frac{\partial^2 E}{\partial \alpha \partial S} < 0; \tag{1.3}$$

$$h = \frac{\partial^2 E}{\partial \alpha \partial \beta} - \frac{\partial^2 E}{\partial \alpha^2} - \frac{\partial^2 E}{\partial \alpha \partial S}\left(\frac{\partial E}{\partial \beta} - \frac{\partial E}{\partial \alpha}\right)\Big/\frac{\partial E}{\partial S} < 0; \tag{1.4}$$

$$q = \frac{\partial^3 E}{\partial \alpha^3} - 3\frac{\partial^2 E}{\partial \alpha^2} + 2\frac{\partial E}{\partial \alpha} < 0, \quad a^2 = c^2 + \frac{2}{3}\left(\frac{\partial^2 E}{\partial \alpha \partial \beta} - \frac{\partial^2 E}{\partial \alpha^2}\right) > 0, \tag{1.5}$$

as well as the inequalities obtained from (1.2)-(1.5) by cyclic substitution of indices.

§2. One-Dimensional Equations

The system of differential equations describing the motion of a viscoelastic medium parallel to the x axis in space (x, y, z) has the form

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